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Discrete Mathematics 307 (2007) 280–284

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Note

The largest eigenvalue of unicyclic graphs☆

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Received 14 September 2005; received in revised form 20 May 2006; accepted 25 June 2006
Available online 17 August 2006

Abstract

Let G be a simple graph. Let $\lambda_1(G)$ and $\mu_1(G)$ denote the largest eigenvalue of the adjacency matrix and the Laplacian matrix of G, respectively. Let Δ denote the largest vertex degree. If G has just one cycle, then

$$\lambda_1(G) \leqslant 2\sqrt{\Delta - 1}.$$

The equality holds if and only if $G \cong C_n$.

And

$$\mu_1(G) \leqslant \Delta + 2\sqrt{\Delta - 1}.$$

The equality holds if and only if $G \cong C_n$, n is even.
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MSC: 05C50; 05C05

Keywords: Unicyclic graph; Adjacency matrix; Laplacian matrix; Largest eigenvalue

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with $V = \{1, 2, \ldots, n\}$. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Both $A(G)$ and $L(G)$ are real symmetric matrices. Let $\lambda_1(G)$ and $\mu_1(G)$ be the largest eigenvalue of $A(G)$ and $L(G)$, respectively. Let d_i denote the degree of $i \in V$ and let Δ denote the largest vertex degree of G.

Let $\mathscr{T}$ be a tree with largest vertex degree Δ, in [2], Godsil proved that

$$\lambda_1(\mathscr{T}) < 2\sqrt{\Delta - 1}. \tag{1}$$

In [4, Theorem 1, p. 36], Stevanović proved (1) again in a different way and proved that

$$\mu_1(\mathscr{T}) < \Delta + 2\sqrt{\Delta - 1}. \tag{2}$$

In this paper, we prove that if G has just one cycle, then

$$\lambda_1(G) \leqslant 2\sqrt{\Delta - 1}.$$

☆ Supported by the Key Project of Chinese Ministry of Education No. 205169.
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doi:10.1016/j.disc.2006.06.023

The equality holds if and only if $G \cong C_n$.

And

$$\mu_1(G) \leqslant \Delta + 2\sqrt{\Delta - 1}.$$

The equality holds if and only if $G \cong C_n$, n is even.

For other undefined notions, we refer the reader to [1].

2. Main results

Definition. Let $\mathscr{T}$ be a tree, if x and y are nonadjacent vertices of $\mathscr{T}$, then $\mathscr{T} + xy$ is obtained from $\mathscr{T}$ by joining x to y. $\mathscr{T} + xy$ just contains one cycle. We call $\mathscr{T} + xy$ an unicyclic graph.

Let C_n denote a cycle of length n, it is well known $\lambda_1(C_n) = 2$ and $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})^{\mathrm{T}}$ is a positive unit eigenvector of $A(C_n)$ corresponding to $\lambda_1(C_n) = 2$.

Theorem 1. *If G is a unicyclic graph and Δ is the maximum vertex degree, then*

$$\lambda_1(G) \leqslant 2\sqrt{\Delta - 1}, \tag{3}$$

where $\lambda_1(G)$ is the maximum adjacency eigenvalue of G. The equality holds if and only if $G \cong C_n$.

Proof. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a positive unit eigenvector of $A(G)$ corresponding to $\lambda_1 = \lambda_1(G)$, let x_i correspond to vertex $i \in V$. We have that

$$\lambda_1 = \lambda_1 \|x\|^2 = \lambda_1 \sum_{i \in V} x_i^2 = \sum_{i \in V} x_i \sum_{\{j:\{i,j\} \in E\}} x_j = 2 \sum_{\{i,j\} \in E} x_i x_j.$$

Since G is a unicyclic graph, there will exist a cycle as a induced subgraph of G, it is denoted by C_k. The vertices of C_k are denoted by $1, 2, \ldots, k$. For C_k, we give an orientation $i \to i + 1$, $i = 1, 2, \ldots, k$, $k + 1 \equiv 1 \pmod k$. The graph $G - E(C_k)$ consists of k trees, it is denoted by $\mathscr{T}_i$, $i = 1, 2, \ldots, k$. We give an orientation to each edge in $\mathscr{T}_i$, such that there exists a directed path from every pendant vertex of $\mathscr{T}_i$ to the vertex $i \in V(C_k)$. Thus we get a digraph $\overrightarrow{G}$.

Since G has just one cycle, we have $|E| = |V|$. For the digraph $\overrightarrow{G}$, let

$$\sum_{\{i,j\} \in E} x_i x_j = x_1 x_{i_1} + x_2 x_{i_2} + \cdots + x_n x_{i_n}, \tag{4}$$

where $i_j \in V(G)$ and the $x_i x_{i_j}$ corresponding to the arc (i, i_j) of $\overrightarrow{G}$. For $\forall i \in V(\overrightarrow{G})$, the outdegree $d^+(i) = 1$ and the indegree $d^-(i) = d_i - 1$. Thus

$$\sum_{j=1}^{n} x_{i_j} = \sum_{i=1}^{n} (d_i - 1) x_i.$$

Similarly, we have

$$\sum_{j=1}^{n} x_{i_j}^2 = \sum_{i=1}^{n} (d_i - 1) x_i^2.$$

From Cauchy–Schwarz inequality applied to (4), it follows that

$$\begin{aligned}\sum_{\{i,j\}\in E} x_i x_j &\leqslant \sqrt{\sum_{i\in V} x_i^2}\sqrt{\sum_{j=1}^{n} x_{i_j}^2}\\ &= \sqrt{1}\sqrt{\sum_{i\in V}(d_i-1)x_i^2}\\ &\leqslant \sqrt{(\Delta-1)\sum_{i\in V}x_i^2} = \sqrt{\Delta-1}.\end{aligned}$$

Thus

$$\lambda_1(G) \leqslant 2\sqrt{\Delta-1}.$$

If the equality holds, then

$$\sum_{i\in V}(d_i-1)x_i^2 = (\Delta-1).$$

And thus

$$d_i = \Delta, \quad i = 1, 2, \ldots, n.$$

Since

$$\sum_{i\in V} d_i = n\Delta = 2|E| = 2|V| = 2n,$$

therefore $\Delta = 2$.

Conversely, if $G \cong C_n$, as it is well known $\Delta = 2$ and $\lambda_1(G) = \Delta$. $\square$

We use already mentioned fact that if H is a subgraph of G, then $\lambda_1(H) \leqslant \lambda_1(G)$. If G is a unicyclic graph, then T is a subgraph of G. Thus we have that:

Corollary 2 (*Godsil [2], Stevanović [4]*). *Let $\mathcal{T}$ be a tree with largest vertex degree Δ, then*

$$\lambda_1(\mathcal{T}) < 2\sqrt{\Delta-1}.$$

Let C_n be a cycle of length n and n is even, then $\mu_1(G) = 4$ and $(1/\sqrt{n}, -1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n})$ is a unit eigenvector of $L(C_n)$ corresponding to $\mu_1(G) = 4$.

The result can be obtained by a simple calculation.

Let G be a graph and let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G.

Lemma 3 (*Guo and Tan [3, Lemma 1, p. 72]*). *The Laplacian eigenvalue of G and $G' = G + e$ interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

Theorem 4. *If G is a unicyclic graph and Δ is the maximum vertex degree, then*

$$\mu_1(G) \leqslant \Delta + 2\sqrt{\Delta-1}, \tag{5}$$

where $\mu_1(G)$ is the maximum Laplacian eigenvalue of G. The equality holds if and only if $G \cong C_n$, n is even.

Proof. Let $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ be a unit eigenvector of $L(G)$ corresponding to $\mu_1 = \mu_1(G)$, let y_i correspond to vertex $i \in V$. Let $A = A(G)$, $D = D(G)$, $L = L(G)$. We have that

$$\begin{aligned}\mu_1 &= \mu_1 \|y\|^2 = \mu_1 y^{\mathrm{T}} y = y^{\mathrm{T}} \mu_1 y = y^{\mathrm{T}} L y \\ &= y^{\mathrm{T}}(D-A)y = y^{\mathrm{T}} D y - y^{\mathrm{T}} A y \\ &= \sum_{i\in V} d_i y_i^2 - \sum_{i\in V} y_i \sum_{\{j:\{i,j\}\in E\}} y_j \\ &= \sum_{i\in V} d_i y_i^2 - 2 \sum_{\{i,j\}\in E} y_i y_j.\end{aligned}$$

Since $\mu_1(G) > 0$, thus

$$\begin{aligned}\mu_1(G) = |\mu_1(G)| &= \left| \sum_{i\in V} d_i y_i^2 - 2 \sum_{\{i,j\}\in E} y_i y_j \right| \\ &\leqslant \left| \sum_{i\in V} d_i y_i^2 \right| + \left| 2 \sum_{\{i,j\}\in E} y_i y_j \right| \\ &\leqslant \Delta \sum_{i\in V} y_i^2 + 2 \sum_{\{i,j\}\in E} |y_i y_j| \\ &\leqslant \Delta + 2\sqrt{\Delta - 1} \quad \text{(similarly as in Theorem 1).}\end{aligned}$$

If the equality holds, then

$$\sum_{i\in V} d_i y_i^2 = \Delta$$

and

$$- \sum_{\{i,j\}\in E} y_i y_j = \sum_{\{i,j\}\in E} |y_i y_j| = \sqrt{\Delta - 1}. \tag{6}$$

We have that $d_i = \Delta$, $i = 1, 2, \ldots, n$. And by Theorem 1 we have that $\Delta = 2$. That is, $G \cong C_n$.

For the C_n, let

$$\sum_{\{i,j\}\in E} y_i y_j = y_1 y_2 + y_2 y_3 + \cdots + y_{n-1} y_n + y_n y_1.$$

By (6), we have that $y_i y_{i+1} < 0$, $i = 1, 2, \ldots, n-1$, and $y_n y_1 < 0$. Thus n is even.

Conversely, if $G \cong C_n$, n is even. Then $\Delta = 2$ and

$$\mu_1(C_n) = \Delta + 2\sqrt{\Delta - 1}. \qquad \square$$

By Lemma 3, we have that:

Corollary 5 (*Stevanović [4]*). *Let $\mathscr{T}$ be a tree with largest vertex degree Δ, then*

$$\mu_1(\mathscr{T}) < \Delta + 2\sqrt{\Delta - 1}.$$

Acknowledgments

The author is grateful to the reviewers for their valuable comments and suggestions.

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